The regular representation, Zhu's $A(V)$-theory and induced modules

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Abstract

The regular representation is related to Zhu's $A(V)$-theory and an induced module from an $A(V)$-module to a V-module is defined in terms of the regular representation. As an application, a new proof of Frenkel and Zhu's fusion rule theorem is obtained.

1 Introduction

In a remarkable paper [Z], Zhu constructed among other things an associative algebra $A(V)$ for each vertex operator algebra V and established a one-to-one correspondence between the set of equivalence classes of irreducible $A(V)$-modules and the set of equivalence classes of lowest weight irreducible generalized V-modules. With this one-to-one correspondence, the classification of irreducible V-modules is reduced to the classification of irreducible $A(V)$-modules. In [FZ], Zhu's $A(V)$-theory was extended further to determine fusion rules by using $A(V)$-modules and bimodules associated to V-modules. Since Zhu had developed his $A(V)$-theory, there have been many applications and generalizations (see for examples [A1-2], [DLM1-4], [DMZ], [DN1-3], [FZ], [KW], [W]). In Zhu's one-to-one correspondence, the functor from a weak V-module to an $A(V)$-module is a restriction with respect to both the space and the algebra, and the functor from an $A(V)$-module to a (weak) V-module is, to a certain extent, analogous to the induction functor in group theory.

In Lie group theory, for a Lie group G and a subgroup H, the induced G-module from an H-module U is defined (cf. [Ki]) to be

$$\mathrm{Ind}_H^G U = \{f : G \to U \mid f(hg) = hf(g) \quad \text{for } h \in H,\ g \in G\},$$

where $(gf)(g') = f(g'g)$ for $g, g' \in G,\ f \in \mathrm{Ind}_G^H U$. The construction of the induced module can be explained as follows: First, $L^2(G)$ or $C^0(G)$ is (naturally) a $G \times G$-module. (Certain $G\times G$-submodules are the modules affording the regular representation of G.) More generally, for any (finite-dimensional) vector space U, the space $C^o(G, U)$ of continuous functions from G to U is a $G \times G$-module. Second, the subspace $\mathrm{Ind}_H^G(U)$ of (left) H-invariant functions from G to U is a G-submodule of $C^0(G, U)$ viewed as a G-module through the identification $G = G \times 1$.

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In [Li3] we defined regular representations of vertex operator algebras and established certain results. More specifically, for a vertex operator algebra V and a nonzero complex number z, we constructed a (weak) $V \otimes V$-module $\mathcal{D}_{P(z)}(V)$ out of the full dual space V^* of V, and we obtained certain results of Peter-Weyl type. Note that unlike in group theory, there is no natural $V \otimes V$-module structure on V^*. In view of this, $\mathcal{D}_{P(z)}(V)$ in a sense plays the role of $C^0(G)$.

The main purpose of this paper is to relate Zhu's $A(V)$-theory to the regular representation in the spirit of the induced module theory for a Lie group. First, for a vector space U, we construct a (weak) $V \otimes V$-module $\mathcal{D}_{P(z)}(V, U)$, a subspace of $\mathrm{Hom}(V, U)$, which plays the role of $C^0(G, U)$. Note that in Zhu's $A(V)$-theory, $A(V)$ is not a subalgebra of V in the usual sense and $A(V)$ does not naturally act on the whole space of a (weak) V-module. In view of this, for an $A(V)$-module U, it does not make sense to consider $A(V)$-invariant functions from V to U. On the other hand, given a (weak) V-module W, there is a canonical $A(V)$-bimodule $A(W)$ [FZ], constructed as a quotient space of W just as $A(V)$ is a quotient space of V (see Section 3 for the definition); and there is an $A(V)$-module $\Omega(W)$, a subspace of W. By definition, $\Omega(W)$ consists of those w such that $v_n w = 0$ for homogeneous $v \in V$ and for $n \geq \mathrm{wt} v$. (Of course, $\Omega(W)$ can also be considered as the invariant space with respect to a certain Lie algebra.) In the case that W is a lowest weight irreducible generalized V-module, $\Omega(W)$ is the lowest weight subspace.

We here define an induced module using the following restriction-expansion strategy. Since $A(V)$ is a quotient space of V, any linear function from $A(V)$ to U lifts to a linear function from V to U. Then we first restrict ourselves to linear functions from V to U, which are lifted from linear functions from $A(V)$ to U, or simply just linear functions from $A(V)$ to U. Now, it makes perfect sense to consider (left) $A(V)$-invariant functions from $A(V)$ to U. It is a classical fact that the space $\mathrm{Hom}(A(V), U)$ of linear functions from $A(V)$ to U is a natural $A(V)$-module containing the space $\mathrm{Hom}_{A(V)}(A(V), U)$ of $A(V)$-invariant linear functions from $A(V)$ to U as a submodule. Of course, $\mathrm{Hom}_{A(V)}(A(V), U)$ is canonically isomorphic to U. On the other hand, it is shown (Proposition 3.8, Theorem 3.9) that $\mathrm{Hom}(A(V), U)$ is a subspace of $\mathcal{D}_{P(-1)}(V, U)$, moreover $\mathrm{Hom}(A(V), U)$ and $\Omega(\mathcal{D}_{P(-1)}(V, U))$ ($\subset \mathrm{Hom}(V, U)$) coincide as natural $A(V) \otimes A(V)$-modules. To summarize, we have the following information:

$$U = \mathrm{Hom}_{A(V)}(A(V), U) \subset \mathrm{Hom}(A(V), U) = \Omega(\mathcal{D}_{P(-1)}(V, U)). \tag{1.1}$$

Then we define the induced module $\mathrm{Ind}_{A(V)}^{V} U$ to be the submodule of $\mathcal{D}_{P(-1)}(V, U)$ generated by $\mathrm{Hom}_{A(V)}(A(V), U)$ $(= U)$ under the action of $V \otimes \mathbb{C}$.

Note that the results of [Li3] were more general than what we needed for regular representations. For any weak V-module W, a weak $V \otimes V$-module $\mathcal{D}_{P(z)}(W)$ was constructed and it was proved that the fusion rule of type $\binom{W'}{W_1 W_2}$ is equal to

$$\dim \mathrm{Hom}_{V \otimes V}(W_1 \otimes W_2, \mathcal{D}_{P(-1)}(W))$$

for generalized V-modules W, W_1, W_2. Furthermore, if W_1 and W_2 are lowest weight generalized V-modules, it was shown (Corollary 4.6, [Li3]) that the fusion rule of type

$\binom{W'}{W_1W_2}$ is equal to

$$\dim \mathrm{Hom}_{A(V)\otimes A(V)}(W_1(0)\otimes W_2(0), \Omega(\mathcal{D}_{P(-1)}(W))),$$

where $W_1(0)$ and $W_2(0)$ are the corresponding lowest weight subspaces. It is proved (Proposition 3.8 and Theorem 3.9) that $\Omega(\mathcal{D}_{P(-1)}(W))$ and $A(W)^*$ coincide as natural $A(V)\otimes A(V)$-modules for any weak V-module W. Using these results, we obtain a new proof of Frenkel and Zhu's fusion rule theorem[2] which asserts that the fusion rule of type $\binom{W_2}{WW_1}$ for irreducible V-modules W, W_1, W_2 is equal to

$$\dim \mathrm{Hom}_{A(V)}(A(W)\otimes_{A(V)} W_1(0), W_2(0))$$

under a certain condition (Corollary 4.17).

In [DLin], an induced module theory for a vertex operator algebra with respect to a vertex operator subalgebra was established. Let V_1 be a vertex operator subalgebra of V and let U be an irreducible V_1-module. In general, U could lift to either a V-module or a so-called twisted V-module by an automorphism of V, but not both. (A V-module is a twisted module corresponding to the identity automorphism.) In this regard, this theory is quite different from and more complicated than the classical theory. We hope to study Dong-Lin's induced module theory in terms of regular representations later.

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The paper is organized as follows: In Section 2, we review the construction of the weak $V\otimes V$-module $\mathcal{D}_{P(z)}(W)$ and the main results, and then construct a weak $V\otimes V$-module $\mathcal{D}_{P(z)}(W,U)$. In Section 3, we identify $\mathrm{Hom}(A(W),U)$ with $\Omega(\mathcal{D}_{P(z)}(W,U))$ as natural $A(V)\otimes A(V)$-modules, and we define the induced V-module $\mathrm{Ind}_{A(V)}^{V}U$ for a given $A(V)$-module U. In Section 4, we give a new proof of the Frenkel and Zhu's fusion rule theorem.

2 Weak $V\otimes V$-modules $\mathcal{D}_{P(z)}(W)$ and $\mathcal{D}_{P(z)}(W,U)$

In this section we shall first review the construction of the weak $V\otimes V$-module $\mathcal{D}_{P(z)}(W)$ and the main results from [Li3], and then construct a (weak) $V\otimes V$-module $\mathcal{D}_{P(z)}(W,U)$ as a generalization.

We use standard definitions and notations as given in [FLM] and [FHL]. A vertex operator algebra is denoted by V, or by $(V, Y, \mathbf{1}, \omega)$ with more information, where $\mathbf{1}$ is the vacuum vector and ω is the Virasoro element. We also use the notion of weak module as defined in [DLM2]—A weak module satisfies all the axioms given in [FLM] and [FHL] for the notion of a module except that no grading is required.

[2]The original theorem [FZ] was corrected in [Li1-2] (see Corollary 4.17 below).

We typically use letters $x, y, x_1, x_2, \dots$ for mutually commuting formal variables and $z, z_0, \dots$ for complex numbers. For a vector space U, $U[[x, x^{-1}]]$ is the vector space of all (doubly infinite) formal series with coefficients in U and $U((x))$ is the space of formal Laurent series. Sometimes we also use $U[x, x^{-1}]]$ for $U((x^{-1}))$. We emphasize the following standard formal variable convention:

$$(x_1 - x_2)^n = \sum_{i \geq 0} (-1)^i \binom{n}{i} x_1^{n-i} x_2^i, \tag{2.1}$$

$$(x - z)^n = \sum_{i \geq 0} (-z)^i \binom{n}{i} x^{n-i}, \tag{2.2}$$

$$(z - x)^n = \sum_{i \geq 0} (-1)^i z^{n-i} \binom{n}{i} x^i \tag{2.3}$$

for $n \in \mathbb{Z}, \ z \in \mathbb{C}^\times$.

Recall the following simple result from [Li3]:

Lemma 2.1 *Let U be a vector space, U_1 a subspace and let*

$$f(x) = \sum_{n \in \mathbb{Z}} f_n x^{-n-1} \in U[[x, x^{-1}]], \quad g(x) = \sum_{n \in \mathbb{Z}} g_n x^{-n-1} \in U_1[[x, x^{-1}]]. \tag{2.4}$$

Suppose that either $f(x) \in U((x))$ or $f(x) \in U((x^{-1}))$ and that there exist $k \in \mathbb{N}$ and $z \in \mathbb{C}^\times$ such that

$$(x - z)^k f(x) = (x - z)^k g(x). \tag{2.5}$$

Then for $n \in \mathbb{Z}$,

$$f_n \in \text{linear span } \{g_m \mid m \geq n\} \tag{2.6}$$

if $f(x) \in U((x))$ and

$$f_n \in \text{linear span } \{g_m \mid m \leq n\} \tag{2.7}$$

if $f(x) \in U((x^{-1}))$. In particular, $f(x) \in U_1[[x, x^{-1}]]$.

For vector spaces U_1, U_2, a linear map $f \in \mathrm{Hom}(U_1, U_2)$ extends canonically to a linear map from $U_1[[x, x^{-1}]]$ to $U_2[[x, x^{-1}]]$. We shall use this canonical extension without any comments.

Let V be a vertex operator algebra. For $v \in V$, we set (cf. [FHL], [HL1])

$$Y^o(v, x) = Y(e^{xL(1)}(-x^{-2})^{L(0)} v, x^{-1}). \tag{2.8}$$

For a weak V-module W, $Y^o(v, x)$ lies in $\mathrm{Hom}(W, W[x, x^{-1}]])$ because $e^{xL(1)}(-x^{-2})^{L(0)} v \in V[x, x^{-1}]$ and $Y(u, x^{-1})w \in W[x, x^{-1}]]$ for $u \in V, \ w \in W$. More generally, for any complex number z_0, $Y^o(v, x + z_0)$ lies in $\mathrm{Hom}(W, W[x, x^{-1}]])$, where by definition

$$Y^o(v, x + z_0)w = (Y^o(v, y)w)|_{y = x + z_0} \tag{2.9}$$

for $w \in W$. Let W be a weak V-module and let U be a vector space, e.g., $U = \mathbb{C}$. For $v \in V,\ f \in \mathrm{Hom}(W, U)$, the compositions $fY^o(v, x)$ and $fY^o(v, x + z_0)$ for any complex number z_0 are elements of $(\mathrm{Hom}(W, U))[[x, x^{-1}]]$.

Now let us review the main definitions and results about $\mathcal{D}_{P(z)}(W)$ from [Li3].

Definition 2.2 [Li3] Let V be a vertex operator algebra, W a weak V-module and z a nonzero complex number. Define $\mathcal{D}_{P(z)}(W)$ to be the subspace of W^*, consisting of those α such that for each $v \in V$, there exist $k, l \in \mathbb{N}$ such that for $w \in W$,

$$x^l (x - z)^k \langle \alpha, Y^o(v, x)w\rangle \in \mathbb{C}[x], \tag{2.10}$$

or what is equivalent, the series $\langle \alpha, Y^o(v, x)w\rangle$, an element of $\mathbb{C}[x, x^{-1}]]$, absolutely converges in the domain $|x| > |z|$ to a rational function of the form $x^{-l}(x - z)^{-k} g(x)$, where $g(x) \in \mathbb{C}[x]$.

The following is an obvious characterization for α lying in $\mathcal{D}_{P(z)}(W)$ without involving matrix-coefficients.

Lemma 2.3 [Li3] *Let W, z be given as before and let $\alpha \in W^*$. Then $\alpha \in \mathcal{D}_{P(z)}(W)$ if and only if for $v \in V$, there exist $k, l \in \mathbb{N}$ such that*

$$x^l (x - z)^k \alpha Y^o(v, x) \in W^*[[x]], \tag{2.11}$$

or equivalently, if and only if for $v \in V$, there exists $k \in \mathbb{N}$ such that

$$(x - z)^k \alpha Y^o(v, x) \in W^*((x)). \tag{2.12}$$

Let $\mathbb{C}(x)$ be the algebra of rational functions of x. The ι-maps $\iota_{x;0}$ and $\iota_{x;\infty}$ from $\mathbb{C}(x)$ to $\mathbb{C}[[x, x^{-1}]]$ are defined as follows: for any rational function $f(x)$, $\iota_{x;0} f(x)$ is the Laurent series expansion of $f(x)$ at $x = 0$ and $\iota_{x;\infty} f(x)$ is the Laurent series expansion of $f(x)$ at $x = \infty$. These are injective $\mathbb{C}[x, x^{-1}]$-linear maps. In terms of the formal variable convention, we have

$$\iota_{x;0}\left((x - z)^n f(x)\right) = (-z + x)^n \iota_{x;0} f(x), \tag{2.13}$$
$$\iota_{x;\infty}\left((x - z)^n f(x)\right) = (x - z)^n \iota_{x;\infty} f(x) \tag{2.14}$$

for $n \in \mathbb{Z},\ z \in \mathbb{C}^\times,\ f(x) \in \mathbb{C}(x)$.

From the definition, for $\alpha \in \mathcal{D}_{P(z)}(W),\ v \in V,\ w \in W$, $\langle \alpha, Y^o(v, x)w\rangle$ lies in the range of $\iota_{x;\infty}$. Then $\iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x)w\rangle$ is a well defined element of $\mathbb{C}(x)$.

Definition 2.4 [Li3] For $v \in V$, $\alpha \in \mathcal{D}_{P(z)}(W)$, we define

$$Y^L_{P(z)}(v, x)\alpha, \quad Y^R_{P(z)}(v, x)\alpha \in W^*[[x, x^{-1}]]$$

by

$$\langle Y^L_{P(z)}(v, x)\alpha, w\rangle = \iota_{x;0}\left(\iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x + z)w\rangle\right) \tag{2.15}$$
$$\langle Y^R_{P(z)}(v, x)\alpha, w\rangle = \iota_{x;0}\iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x)w\rangle \tag{2.16}$$

for $w \in W$.

Lemma 2.5 [Li3] *Let $v \in V$, $\alpha \in \mathcal{D}_{P(z)}(W)$. Then*

$$(-z+x)^k Y^R_{P(z)}(v,x)\alpha = (x-z)^k \alpha Y^o(v,x), \tag{2.17}$$
$$(z+x)^l Y^L_{P(z)}(v,x)\alpha = (x+z)^l \alpha Y^o(v,x+z), \tag{2.18}$$

where k and l are any pair of nonnegative integers such that (2.11) holds.

We have ([Li3], Proposition 3.24):

Proposition 2.6 [Li3] *Let W be a weak V-module and let z be a nonzero complex number. Then*

$$Y^L_{P(z)}(v,x)\alpha, \quad Y^R_{P(z)}(v,x)\alpha \in (\mathcal{D}_{P(z)}(W))((x)) \tag{2.19}$$

for $v \in V$, $\alpha \in \mathcal{D}_{P(z)}(W)$. Furthermore,

$$Y^L_{P(z)}(u,x_1)Y^R_{P(z)}(v,x_2) = Y^R_{P(z)}(v,x_2)Y^L_{P(z)}(u,x_1) \tag{2.20}$$

on $\mathcal{D}_{P(z)}(W)$ for $u, v \in V$.

In view of Proposition 2.6, $Y^L_{P(z)}$ and $Y^R_{P(z)}$ give rise to a well defined linear map

$$Y_{P(z)} = Y^L_{P(z)} \otimes Y^R_{P(z)} : V \otimes V \to \Big(\mathrm{End}\, \mathcal{D}_{P(z)}(W)\Big)[[x,x^{-1}]]. \tag{2.21}$$

Then we have ([Li3], Theorem 3.17, Propositions 3.21 and 3.24 and Theorem 3.25):

Theorem 2.7 [Li3] *Let W be a weak V-module and let z be a nonzero complex number. Then the pairs $(\mathcal{D}_{P(z)}(W), Y^L_{P(z)})$ and $(\mathcal{D}_{P(z)}(W), Y^R_{P(z)})$ carry the structure of a weak V-module and the pair $(\mathcal{D}_{P(z)}(W), Y_{P(z)})$ carries the structure of a weak $V \otimes V$-module.*

For a $\mathbb{C}$-graded vector space $M = \coprod_{h\in\mathbb{C}} M_{(h)}$, following [HL1] we define the formal completion

$$\overline{M} = \prod_{h\in\mathbb{C}} M_{(h)}. \tag{2.22}$$

Recall from [FHL] that $M' = \coprod_{h\in\mathbb{C}} M^*_{(h)}$. Then

$$\overline{M'} = M^*. \tag{2.23}$$

We shall need the following notions. A *generalized* V-module [HL1] is a weak V-module on which $L(0)$ semisimply acts. Then for a generalized V-module W we have the $L(0)$-eigenspace decomposition: $W = \coprod_{h\in\mathbb{C}} W_{(h)}$. Thus, a generalized V-module satisfies all the axioms defining the notion of a V-module ([FLM], [FHL]) except the two grading restrictions on the homogeneous subspaces. If a generalized V-module furthermore satisfies the lower truncation condition (one of the two grading restrictions), we call it a *lower truncated* generalized module [H1].

Following [HL1], we choose a branch $\log z$ of the log function so that

$$\log z = \log |z| + i \arg z \quad \text{with} \quad 0 \leq \arg z < 2\pi, \tag{2.24}$$

and arbitrary values of the log function will be denoted by

$$l_p(z) = \log z + 2p\pi i \tag{2.25}$$

for $p \in \mathbb{Z}$.

Let W, W_1 and W_2 be generalized V-modules and let $\mathcal{Y}$ be an intertwining operator of type $\binom{W'}{W_1 W_2}$. For $w_{(1)} \in W_1,\ w_{(2)} \in W_2$, we set [HL1]

$$\mathcal{Y}(w_{(1)}, e^{l_p(z)})w_{(2)} = \Big(\mathcal{Y}(w_{(1)}, x)w_{(2)}\Big)\,|_{x^h = e^{hl_p(z)}, h\in\mathbb{C}} \in \overline{W'}\ (= W^*). \tag{2.26}$$

Note that $\mathcal{Y}(w_{(1)}, x)w_{(2)}$ in general involves non-integral, even complex powers of x. We have ([Li3], Theorem 4.5):

Proposition 2.8 [Li3] *Let W, W_1 and W_2 be generalized V-modules, $\mathcal{Y}$ an intertwining operator of type $\binom{W'}{W_1 W_2}$ and let $p \in \mathbb{Z}$. Then*

$$\mathcal{Y}(w_{(1)}, e^{l_p(z)})w_{(2)} \in \mathcal{D}_{P(z)}(W) \tag{2.27}$$

for $w_{(1)} \in W_1,\ w_{(2)} \in W_2$.

In view of Proposition 2.8, for an intertwining operator $\mathcal{Y}$ of type $\binom{W'}{W_1 W_2}$ we have a linear map

$$\begin{aligned} F^{P(z)}_{\mathcal{Y},p}: \quad & W_1 \otimes W_2 \to \mathcal{D}_{P(z)}(W) \\ & (w_{(1)}, w_{(2)}) \mapsto F^{P(z)}_{\mathcal{Y},p}(w_{(1)} \otimes w_{(2)}) = \mathcal{Y}(w_{(1)}, e^{l_p(z)})w_{(2)} \end{aligned} \tag{2.28}$$

for $w_{(1)} \in W_1,\ w_{(2)} \in W_2$.

For generalized V-modules W, W_1 and W_2, following [HL1] we denote by $\mathcal{V}^{W'}_{W_1 W_2}$ the space of intertwining operators of type $\binom{W'}{W_1 W_2}$. Then we have ([Li3], Corollary 4.6):

Theorem 2.9 [Li3] *Let W, W_1 and W_2 be lower truncated generalized V-modules, let z be a nonzero complex number and let $p \in \mathbb{Z}$. Then the linear map*

$$\begin{aligned} F_p[P(z)]^{W'}_{W_1 W_2}: \quad & \mathcal{V}^{W'}_{W_1 W_2} \to \mathrm{Hom}_{V\otimes V}(W_1 \otimes W_2, \mathcal{D}_{P(z)}(W)) \\ & \mathcal{Y} \mapsto F^{P(z)}_{\mathcal{Y},p} \end{aligned} \tag{2.29}$$

is a linear isomorphism.

Next, we shall generalize the notion of $\mathcal{D}_{P(z)}(W)$ by incorporating a vector space U.

Definition 2.10 Let W be a weak V-module, U a vector space and z a nonzero complex number. Define $\mathcal{D}_{P(z)}(W, U)$ to be the subset of $\mathrm{Hom}(W, U)$, consisting of each f such that for $v \in V$, there exist $k, l \in \mathbb{N}$ such that

$$(x-z)^k x^l \langle u^*, fY^o(v, x)w\rangle \in \mathbb{C}[x] \tag{2.30}$$

for all $u^* \in U^*,\ w \in W$, or what is equivalent, for all $u^* \in U^*,\ w \in W$, the formal series

$$\langle u^*, fY^o(v, x)w\rangle,$$

an element of $\mathbb{C}[x, x^{-1}]]$, absolutely converges in the domain $|x| > |z|$ to a rational function of the form $x^{-l}(x-z)^{-k}g(x)$ for $g(x) \in \mathbb{C}[x]$.

Clearly, $\mathcal{D}_{P(z)}(W, U)$ is a subspace of $\mathrm{Hom}(W, U)$. When $U = \mathbb{C}$, $\mathcal{D}_{P(z)}(W, \mathbb{C})$ gives us $\mathcal{D}_{P(z)}(W)$.

Lemma 2.11 *Let $f \in \mathrm{Hom}(W, U)$. Then the following statements are equivalent:*
(a) $f \in \mathcal{D}_{P(z)}(W, U)$.
(b) For $v \in V$, there exist $k, l \in \mathbb{N}$ such that

$$(x-z)^k x^l fY^o(v, x) \in (\mathrm{Hom}(W, U))[[x]]. \tag{2.31}$$

(c) For $v \in V$, there exist $k, l \in \mathbb{N}$ such that for each $w \in W$,

$$(x-z)^k x^l fY^o(v, x)w \in U[x]. \tag{2.32}$$

Proof. Clearly, (a) implies (b), and (c) implies (a). Since $Y^o(v, x)w \in W[x, x^{-1}]]$ for $v \in V,\ w \in W$, we see that (b) implies (c). $\square$

Let $v \in V,\ f \in \mathcal{D}_{P(z)}(W, U)$ and let $k, l \in \mathbb{N}$ be such that (2.32) holds. Then by changing variable we get

$$x^k (x+z)^l fY^o(v, x+z)w \in U[x] \tag{2.33}$$

for $w \in W$.

Definition 2.12 Let W, U and z be given as before. For $v \in V,\ f \in \mathcal{D}_{P(z)}(W, U)$, we define two elements $Y^L_{P(z)}(v, x)f$ and $Y^R_{P(z)}(v, x)$ of $(\mathrm{Hom}(W, U))[[x, x^{-1}]]$ by

$$\begin{aligned}(Y^L_{P(z)}(v, x)f)(w) &= (z+x)^{-l}\left((x+z)^l f(Y^o(v, x+z)w)\right) && (2.34)\\ (Y^R_{P(z)}(v, x)f)(w) &= (-z+x)^{-k}\left((x-z)^k f(Y^o(v, x)w)\right) && (2.35)\end{aligned}$$

for $w \in W$, where k, l are any pair of (possibly negative) integers such that (2.32) holds.

First, in view of (2.32) and (2.33), both $(z+x)^{-l}\left((x+z)^l f(Y^o(v,x+z)w)\right)$ and $(-z+x)^{-k}\left((x-z)^k f(Y^o(v,x)w)\right)$ lie in $U((x))$, so that $Y^L_{P(z)}(v,x)f$ and $Y^R_{P(z)}(v,x)f$ make sense. However, we are not allowed to remove the left-right brackets to cancel $(x-z)^k$ or $(x+z)^l$ because of the nonexistence of terms $(z+x)^{-l}f(Y^o(v,x+z)w)$ and $(-z+x)^{-k}f(Y^o(v,x)w)$. Second, they are also well defined, i.e., they are independent of the choice of the pair of integers k,l. Indeed, if k',l' are another pair of integers such that (2.32) holds, say for example, $k\geq k'$, then

$$\begin{aligned}
& (-z+x)^{-k}\left((x-z)^k fY^o(v,x)w\right) \\
= \; & (-z+x)^{-k}\left((x-z)^{k-k'}(x-z)^{k'} fY^o(v,x)w\right) \\
= \; & (-z+x)^{-k}(x-z)^{k-k'}\left((x-z)^{k'} fY^o(v,x)w\right) \\
= \; & (-z+x)^{-k'}\left((x-z)^{k'} fY^o(v,x)w\right).
\end{aligned} \tag{2.36}$$

From definition we immediately have:

Lemma 2.13 *For $v\in V,\ f\in \mathcal{D}_{P(z)}(W,U)$,*

$$(z+x)^l Y^L_{P(z)}(v,x)f = (x+z)^l fY^o(v,x+z), \tag{2.37}$$
$$(-z+x)^k Y^R_{P(z)}(v,x)f = (x-z)^k fY^o(v,x), \tag{2.38}$$

where k,l are any pair of integers such that (2.32) holds. $\square$

In terms of rational functions and the ι-maps we immediately have (cf. [DL], [FHL]):

Lemma 2.14 *For $v\in V,\ f\in \mathcal{D}_{P(z)}(W,U),\ u^*\in U^*,\ w\in W$,*

$$\langle u^*, (Y^L_{P(z)}(v,x)f)(w)\rangle = \iota_{x;0}\iota^{-1}_{x;\infty}\langle u^*, fY^o(v,x+z)w\rangle, \tag{2.39}$$
$$\langle u^*, (Y^R_{P(z)}(v,x)f)(w)\rangle = \iota_{x;0}\iota^{-1}_{x;\infty}\langle u^*, fY^o(v,x)w\rangle. \quad \square \tag{2.40}$$

Let W,U and z be given as before. Consider $U^*\otimes W$ as a weak V-module with the action of V on W. Then in view of Theorem 2.7 we have a weak $V\otimes V$-module $\mathcal{D}_{P(z)}(U^*\otimes W)$. Let $\alpha\in (U^*\otimes W)^*$. Then $\alpha\in \mathcal{D}_{P(z)}(U^*\otimes W)$ if and only if for $v\in V$, there exist $k,l\in\mathbb{N}$ such that

$$(x-z)^k x^l\langle \alpha, u^*\otimes Y^o(v,x)w\rangle \in \mathbb{C}[x] \tag{2.41}$$

for all $u^*\in U^*,\ w\in W$.

Let η be the canonical embedding of $\mathrm{Hom}(W,U)$ into $(U^*\otimes W)^*$, i.e., for $f\in \mathrm{Hom}(W,U)$, $u^*\in U^*,\ w\in W$,

$$\langle \eta(f), u^*\otimes w\rangle = \langle u^*, f(w)\rangle. \tag{2.42}$$

Let $f\in \mathcal{D}_{P(z)}(W,U)$ ($\subset \mathrm{Hom}(W,U)$). For $v\in V$, let $l,k\in\mathbb{N}$ such that

$$(x-z)^k x^l\langle u^*, fY^o(v,x)w\rangle \in \mathbb{C}[x] \tag{2.43}$$

for all $u^* \in U^*,\ w \in W$, that is,

$$(x-z)^k x^l \langle \eta(f), u^* \otimes Y^o(v,x)w \rangle \in \mathbb{C}[x] \tag{2.44}$$

for all $u^* \in U^*,\ w \in W$. Then $\eta(f) \in \mathcal{D}_{P(z)}(U^* \otimes W)$. This proves

$$\eta(\mathcal{D}_{P(z)}(W,U)) \subset \mathcal{D}_{P(z)}(U^* \otimes W). \tag{2.45}$$

On the other hand, let $f \in \mathrm{Hom}(W,U)$. If $\eta(f) \in \mathcal{D}_{P(z)}(U^* \otimes W)$, for $v \in V$, there exist $k,l \in \mathbb{N}$ such that

$$(x-z)^k x^l \langle \eta(f), u^* \otimes Y^o(v,x)w \rangle \in \mathbb{C}[x] \tag{2.46}$$

for all $u^* \in U^*,\ w \in W$. That is,

$$(x-z)^k x^l \langle u^*, f(Y^o(v,x)w) \rangle \in \mathbb{C}[x]. \tag{2.47}$$

Then $f \in \mathcal{D}_{P(z)}(W,U)$. This shows

$$\eta(\mathrm{Hom}(W,U)) \cap \mathcal{D}_{P(z)}(U^* \otimes W) \subset \eta(\mathcal{D}_{P(z)}(W,U)). \tag{2.48}$$

Therefore, we have proved:

Lemma 2.15 *Let W be a weak V-module, U a vector space and z a nonzero complex number. Then*

$$\eta(\mathcal{D}_{P(z)}(W,U)) = \eta(\mathrm{Hom}(W,U)) \cap \mathcal{D}_{P(z)}(U^* \otimes W). \quad \square \tag{2.49}$$

Furthermore, we have:

Proposition 2.16 *Let W be a weak V-module, and let U be a vector space. Then $\eta(\mathcal{D}_{P(z)}(W,U))$ is a weak $V \otimes V$-submodule of $\mathcal{D}_{P(z)}(U^* \otimes W)$. Furthermore,*

$$\eta(Y^L_{P(z)}(v,x)f) = Y^L_{P(z)}(v,x)\eta(f), \quad \eta(Y^R_{P(z)}(v,x)f) = Y^R_{P(z)}(v,x)\eta(f) \tag{2.50}$$

for $v \in V,\ f \in \mathcal{D}_{P(z)}(W,U)$.

Proof. Let $v \in V,\ f \in \mathcal{D}_{P(z)}(W,U)$. Since $\eta(f) \in \mathcal{D}_{P(z)}(U^* \otimes W)$ (Lemma 2.15), by Lemma 2.5 there exist $k,l \in \mathbb{N}$ such that

$$\begin{aligned} (x-z)^k Y^R_{P(z)}(v,x)\eta(f) &= (x-z)^k \eta(f) Y^o(v,x) && (2.51)\\ (x+z)^l Y^L_{P(z)}(v,x)\eta(f) &= (x+z)^l \eta(f) Y^o(v,x+z). && (2.52) \end{aligned}$$

For $u^* \in U^*,\ w \in W$, we have

$$\begin{aligned} \langle \eta(f) Y^o(v,x), u^* \otimes w \rangle &= \langle \eta(f), Y^o(v,x)(u^* \otimes w) \rangle \\ &= \langle \eta(f), u^* \otimes Y^o(v,x)w \rangle \\ &= \langle u^*, f Y^o(v,x)w \rangle \\ &= \langle \eta(f Y^o(v,x)), u^* \otimes w \rangle. && (2.53) \end{aligned}$$

Then

$$\eta(f)Y^o(v,x) = \eta(fY^o(v,x)). \tag{2.54}$$

Consequently,

$$\eta(f)Y^o(v,x) \ \ (= \eta(fY^o(v,x))) \in \eta(\mathrm{Hom}(W,U))[[x,x^{-1}]]. \tag{2.55}$$

Then it follows from Lemma 2.1 and (2.51)-(2.52) that

$$Y^R_{P(z)}(v,x)\eta(f), \ \ Y^L_{P(z)}(v,x)\eta(f) \in \eta(\mathrm{Hom}(W,U))[[x,x^{-1}]],$$

so that from Lemma 2.15,

$$Y^R_{P(z)}(v,x)\eta(f), \ \ Y^L_{P(z)}(v,x)\eta(f) \in \eta(\mathcal{D}_{P(z)}(W,U))[[x,x^{-1}]]. \tag{2.56}$$

This proves that $\eta(\mathcal{D}_{P(z)}(W,U))$ is a weak $V\otimes V$-submodule of $\mathcal{D}_{P(z)}(U^*\otimes W)$.

Let $v\in V,\ f\in \mathcal{D}_{P(z)}(W,U),\ u^*\in U^*,\ w\in W$. Then using Lemma 2.14 we get

$$\begin{aligned}
\langle Y^L_{P(z)}(v,x)\eta(f), u^*\otimes w\rangle &= \iota_{x;0}\iota_{x;\infty}^{-1}\langle \eta(f), u^*\otimes Y^o(v,x+z)w)\rangle\\
&= \iota_{x;0}\iota_{x;\infty}^{-1}\langle u^*, f(Y^o(v,x+z)w)\rangle\\
&= \langle u^*, (Y^L_{P(z)}(v,x)f)w)\rangle\\
&= \langle \eta(Y^L_{P(z)}(v,x)f), u^*\otimes w\rangle.
\end{aligned} \tag{2.57}$$

Thus

$$\eta(Y^L_{P(z)}(v,x)f) = Y^L_{P(z)}(v,x)\eta(f).$$

Similarly we can prove

$$\eta(Y^R_{P(z)}(v,x)f) = Y^R_{P(z)}(v,x)\eta(f).$$

This completes the proof. $\square$

In view of Theorem 2.7 and Proposition 2.16 we immediately have:

Theorem 2.17 *Let W be a weak V-module, U a vector space and z a nonzero complex number. Then the pairs $(\mathcal{D}_{P(z)}(W,U), Y^L_{P(z)})$ and $(\mathcal{D}_{P(z)}(W,U), Y^R_{P(z)})$ carry the structure of a weak $V\otimes V$-module and the actions $Y^L_{P(z)}$ and $Y^R_{P(z)}$ of V on $\mathcal{D}_{P(z)}(W,U)$ commute. Furthermore, set*

$$Y_{P(z)} = Y^L_{P(z)}\otimes Y^R_{P(z)}. \tag{2.58}$$

Then the pair $(\mathcal{D}_{P(z)}(W,U), Y_{P(z)})$ carries the structure of a weak $V\otimes V$-module. $\square$

In view of Proposition 2.16 and (2.54), from ([Li3], Proposition 3.22) we immediately have the following relations among $fY^o(v,x), Y^L(v,x)$ and $Y^R(v,x)f$:

Corollary 2.18 *Let $v\in V,\ f\in \mathcal{D}_{P(z)}(W,U)$. Then*

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x-z}{x_0}\right) fY^o(v,x) - x_0^{-1}\delta\left(\frac{z-x}{-x_0}\right) Y^R_{P(z)}(v,x)f\\
&= z^{-1}\delta\left(\frac{x-x_0}{z}\right) Y^L_{P(z)}(v,x_0)f.
\end{aligned} \tag{2.59}$$

For convenience, from now on we shall drop the "$P(z)$" from the notations $Y^L_{P(z)}$ and $Y^R_{P(z)}$ when there is no confusion.

3 Zhu's $A(V)$-theory and induced module $\mathrm{Ind}_{A(V)}^{V} U$

In this section, given a weak V-module W and a nonzero complex number z, we construct an $A(V)\otimes A(V)$-module $A(W, z)$, generalizing Frenkel and Zhu's notion of $A(W)$, and then we relate $\mathrm{Hom}(A(W, z), U)$ to a canonical subspace of $\mathcal{D}_{P(z)}(W, U)$. Using this connection, we define the induced module $\mathrm{Ind}_{A(V)}^{V} U$ from an $A(V)$-module U.

First we define or review certain notions. A *lowest weight* generalized V-module is a generalized V-module such that $W = \coprod_{n\in\mathbb{N}} W_{(h+n)}$ for some $h \in \mathbb{C}$ and $W_{(h)}$ generates W. Furthermore, if $W \neq 0$, we call the unique h *the lowest weight* of W. An $\mathbb{N}$-*graded* weak V-module [Z] is a weak V-module W together with an $\mathbb{N}$-grading $W = \coprod_{n\in\mathbb{N}} W(n)$ such that

$$v_m W(n) \subset W(n + \mathrm{wt} v - m - 1) \tag{3.1}$$

for homogeneous $v \in V$ and for $m \in \mathbb{Z},\ n \in \mathbb{N}$, where by definition $W(n) = 0$ for $n < 0$. An $\mathbb{N}$-*gradable* weak V-module is a weak V-module W on which there exists an $\mathbb{N}$-grading such that W together the grading becomes an $\mathbb{N}$-graded module. A vertex operator algebra V is said to be *rational* [Z] (cf. [DLM2]) if every $\mathbb{N}$-gradable weak V-module is a direct sum of irreducible $\mathbb{N}$-gradable weak V-modules. There are also different definitions of rationality (see for example [HL1]).

Now we recall Zhu's construction of $A(V)$ and the main results from [Z]. Let V be a vertex operator algebra. Set

$$O(V) = \text{linear span}\{\mathrm{Res}_x x^{-2}(1+x)^{\mathrm{wt} u} Y(u, x)v \ \text{ for homogeneous } u, v \in V\}. \tag{3.2}$$

Note that we *do not* assume that V has the special property that $V = \oplus_{n\geq 0} V_{(n)}$, so that $\mathrm{wt} u$ could be *negative*, hence the formal series $(1 + x)^{\mathrm{wt} u}$ and $(x + 1)^{\mathrm{wt} u}$ may be different.

For homogeneous $u, v \in V$, we define [Z]

$$u * v = \mathrm{Res}_x x^{-1}(1+x)^{\mathrm{wt} u} Y(u, x)v \left(= \sum_{i\geq 0} \binom{\mathrm{wt} u}{i} u_{i-1} v \right). \tag{3.3}$$

Then extend the definition of $*$ on V by linearity. Set

$$A(V) = V/O(V). \tag{3.4}$$

The following is the first of Zhu's theorems in his $A(V)$-theory.

Proposition 3.1 [Z] *Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra. Then the space $O(V)$ is a two-sided ideal of the nonassociative algebra $(V, *)$ and the quotient algebra $A(V)$ $(= V/O(V))$ is an associative algebra with $\mathbf{1}+O(V)$ being the identity element and with $\omega + O(V)$ being a central element. Furthermore, $A(V)$ has an involution (anti-automorphism) θ given by*

$$\theta(v) = e^{L(1)}(-1)^{L(0)} v \quad \textit{for } v \in V. \tag{3.5}$$

Let W be a weak V-module. Following [DLM2] we define

$$\Omega(W)=\{w\in W \mid v_n w=0 \;\text{ for homogeneous } v\in V \text{ and for } n\geq \mathrm{wt}v\}. \tag{3.6}$$

Equivalently, $w\in\Omega(W)$ if and only if $x^{\mathrm{wt}v}Y(v,x)w\in W[[x]]$ for each homogeneous $v\in V$. Then we have ([Z], [DLM2]):

Proposition 3.2 *For any weak V-module W, $\Omega(W)$ is a natural $A(V)$-module with $v+O(V)$ acting on $\Omega(W)$ as $v_{\mathrm{wt}v-1}$ for homogeneous $v\in V$. Furthermore, if $W=\coprod_{n\geq 0}W_{(n+h)}$ is a lowest weight irreducible generalized V-module with $W_{(h)}\neq 0$, then $\Omega(W)=W_{(h)}$ and it is an irreducible $A(V)$-module.*

Let W_1, W_2 be weak V-modules and let ψ be a V-homomorphism from W_1 to W_2. Clearly, $\psi(\Omega(W_1))\subset\Omega(W_2)$ and the restriction $\Omega(\psi):=\psi|_{\Omega(W_1)}$ is an $A(V)$-homomorphism. It is routine to check that Ω is a functor from the category of weak V-modules to the category of $A(V)$-modules. On the other hand, for any $A(V)$-module U Zhu in [Z] constructed a $\mathbb{N}$-graded weak V-module $L(U)$ with $U=L(U)(0)\subset\Omega(L(U))$ (cf. [DLM2]). Now we shall use the generalized regular representation of V on $\mathcal{D}_{P(z)}(V,U)$ to construct such an $\mathbb{N}$-graded weak V-module.

Let W be a weak V-module and let z be a nonzero complex number. Generalizing the definition of $O(W)$ in [FZ], we define $O(W,z)$ to be the subspace of W, linearly spanned by elements

$$\mathrm{Res}_x x^{-2}(1-zx)^{\mathrm{wt}v}Y(v,x)w \tag{3.7}$$

for homogeneous $v\in V$ and for $w\in W$. With this notion, $O(W)=O(W,-1)$. Generalizing Frenkel and Zhu's left and right actions of V on W [FZ] we define

$$v*_{P(z)}w = \mathrm{Res}_x(-z)^{-\mathrm{wt}v}x^{-1}(1-zx)^{\mathrm{wt}v}Y(v,x)w, \tag{3.8}$$
$$w*_{P(z)}v = \mathrm{Res}_x(-z)^{-\mathrm{wt}v}x^{-1}(1-zx)^{\mathrm{wt}v-1}Y(v,x)w \tag{3.9}$$

for homogeneous $v\in V$ and for $w\in W$. Then extend the definitions by linearity. (We recover Frenkel and Zhu's actions when $z=-1$.) In the following we shall show that these generalized actions actually are Frenkel and Zhu's actions of V on W with respect to a *new module* structure.

Lemma 3.3 *Let W be a weak V-module and let z be a nonzero complex number. For $v\in V$, set*

$$Y^{(z)}(v,x)=Y(z^{L(0)}v,zx). \tag{3.10}$$

Then $(W,Y^{(z)})$ carries the structure of a weak V-module. Furthermore, for homogeneous $v\in V$ and for $m,n\in\mathbb{Z}$, we have

$$\mathrm{Res}_x(-z)^{-\mathrm{wt}v}x^m(1-zx)^nY(v,x)=(-z)^{-m-1}\mathrm{Res}_x x^m(1+x)^nY^{(-z^{-1})}(v,x). \tag{3.11}$$

Proof. From [FHL], we have

$$z^{L(0)}Y(v,x)z^{-L(0)} = Y(z^{L(0)}, zx) \tag{3.12}$$

on any generalized V-module (on which $L(0)$ semisimply acts). In particular, this is true on the adjoint module V. If W is a generalized V-module, it follows immediately from (3.12) that $(W, Y^{(z)})$ is a weak V-module and it is isomorphic to (W, Y) through the map $z^{L(0)}$. For a general weak V-module W, replacing (u, v) and (x_0, x_1, x_2) by $(z^{L(0)}u, z^{L(0)}v)$ and (zx_0, zx_1, zx_2) in the Jacobi identity for Y, respectively, then using (3.12) on V we obtain

$$\begin{aligned}
& z^{-1}x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y^{(z)}(u,x_1)Y^{(z)}(v,x_2)\\
- \;& z^{-1}x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y^{(z)}(v,x_2)Y^{(z)}(u,x_1)\\
= \;& z^{-1}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(z^{L(0)}u,zx_0)z^{L(0)}v,zx_2)\\
= \;& z^{-1}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(z^{L(0)}Y(u,x_0)v,zx_2)\\
= \;& z^{-1}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y^{(z)}(Y(u,x_0)v,x_2).
\end{aligned} \tag{3.13}$$

This proves the Jacobi identity for $Y^{(z)}$ while the vacuum property and lower truncation condition clearly hold. The identity (3.11) directly follows from changing variable $y = -z^{-1}x$. $\square$

With Lemma 3.3, generalizations of certain Zhu's theorems [Z], or Frenkel and Zhu's theorems [FZ] will follow immediately. First, we have (cf. [Z]):

Lemma 3.4 *Let W be a weak V-module and let z be a nonzero complex number. Then*

$$\mathrm{Res}_x x^{-n-2}(1-zx)^{\mathrm{wt}v+m}Y(v,x)w \in O(W,z) \tag{3.14}$$

for homogeneous $v \in V$ and for $n \geq m \geq 0$, $w \in W$. $\square$

We also have:

Lemma 3.5 *Let W and z be given as before. Then*

$$\mathrm{Res}_x x^{-n-2}(1-zx)^{\mathrm{wt}v+m}Y(e^{x^{-1}L(1)}v,x)w \in O(W,z) \tag{3.15}$$

for any homogeneous $v \in V$ and for $n \geq m \geq 0$, $w \in W$.

Proof. Notice that $\mathrm{wt}(L(1)^iv) = \mathrm{wt}v - i$ for $i \geq 0$. Then using Lemma 3.4, we get

$$\begin{aligned}
& \mathrm{Res}_x x^{-n-2}(1-zx)^{\mathrm{wt}v+m}Y(e^{x^{-1}L(1)}v,x)w\\
= \;& \sum_{i\geq 0}\mathrm{Res}_x \frac{1}{i!}x^{-n-i-2}(1-zx)^{\mathrm{wt}v+m}Y(L(1)^iv,x)w\\
= \;& \sum_{i\geq 0}\mathrm{Res}_x \frac{1}{i!}x^{-n-i-2}(1-zx)^{(\mathrm{wt}v-i)+i+m}Y(L(1)^iv,x)w \in O(W,z). \quad \square
\end{aligned} \tag{3.16}$$

Set $A(W,z)=W/O(W,z)$. Then $A(W)=A(W,-1)$. Noticing that from Lemma 3.3, the left and right actions $*_{P(z)}$ are exactly the Frenkel and Zhu's left and right actions on the module $(W,Y^{(-z^{-1})})$, we immediately have:

Proposition 3.6 [FZ] *Let W be a weak V-module and let z be a nonzero complex number. Then the left and right actions $*_{P(z)}$ of V on W defined in (3.8) and (3.9) give rise to an $A(V)$-bimodule structure on $A(W,z)$.*

We shall need the following result:

Lemma 3.7 *Let V_1 and V_2 be vertex operator algebras and let E be a weak $V_1 \otimes V_2$-module. Then*

$$\Omega_{V_1\otimes V_2}(E)=\Omega_{V_1}(E)\cap\Omega_{V_2}(E), \tag{3.17}$$

where E is considered as a weak V_1-module and a weak V_2-module in the obvious way.

Proof. Clearly,

$$\Omega_{V_1\otimes V_2}(E)\subset\Omega_{V_1}(E)\cap\Omega_{V_2}(E).$$

On the other hand, since the actions of V_1 and V_2 on E commute,

$$Y(v_{(2)},x)\Omega_{V_1}(E)\subset\Omega_{V_1}(E)[[x,x^{-1}]] \tag{3.18}$$

for $v_{(2)}\in V_2$. Now, let $e\in\Omega_{V_1}(E)\cap\Omega_{V_2}(E)$ and let $v_{(1)}\in V_1,\ v_{(2)}\in V_2$ be homogeneous. Then

$$x^{\mathrm{wt} v_{(2)}}Y(v_{(2)},x)e\in E[[x]]\cap\Omega_{V_1}(E)[[x,x^{-1}]]=\Omega_{V_1}(E)[[x]], \tag{3.19}$$

so that using (3.18) we get

$$x^{(\mathrm{wt} v_{(1)}\otimes v_{(2)})}Y(v_{(1)}\otimes v_{(2)},x)e=x^{\mathrm{wt} v_{(1)}}Y(v_{(1)},x)\left(x^{\mathrm{wt} v_{(2)}}Y(v_{(2)},x)e\right)\in E[[x]]. \tag{3.20}$$

Thus $e\in\Omega_{V_1\otimes V_2}(E)$. This proves

$$\Omega_{V_1}(E)\cap\Omega_{V_2}(E)\subset\Omega_{V_1\otimes V_2}(E)$$

and completes the proof. $\square$

Now let W be a weak V-module, U a vector space and z a nonzero complex number. Consider $\mathrm{Hom}(A(W,z),U)$ naturally as a subspace of $\mathrm{Hom}(W,U)$. Recall that η is the canonical embedding of $\mathrm{Hom}(W,U)$ into $(U^*\otimes W)^*$. Then we have:

Proposition 3.8 *Let W be a weak V-module, U a vector space, and z a nonzero complex number. Then*

$$\begin{aligned} & \mathrm{Hom}(A(W;z),U) \\ = \;& \Omega(\mathcal{D}_{P(z)}(W,U)) && (3.21)\\ = \;& \{f\in\mathrm{Hom}(W,U)\mid x^{\mathrm{wt} v}(x-z)^{\mathrm{wt} v}fY^{o}(v,x)\in(\mathrm{Hom}(W,U))[[x]]\quad \textit{for homogeneous } v\in V\} && (3.22)\end{aligned}$$

Furthermore,

$$(-z+x)^{\mathrm{wt}v}Y^R(v,x)f=(x-z)^{\mathrm{wt}v}fY^o(v,x), \tag{3.23}$$
$$(z+x)^{\mathrm{wt}v}Y^L(v,x)f=(x+z)^{\mathrm{wt}v}fY^o(v,x+z) \tag{3.24}$$

for $f\in \mathrm{Hom}(A(W,z),U)$ *and for homogeneous* $v\in V$.

Proof. Let T be the set defined in the right hand side of (3.22). To prove the first assertion, in the following we shall prove

$$\mathrm{Hom}(A(W,z),U)\subset T\subset \Omega(\mathcal{D}_{P(z)}(W,U))\subset \mathrm{Hom}(A(W,z),U).$$

The second part follows immediately from Lemma 2.13.

Let $f\in \mathrm{Hom}(A(W,z),U)$ ($\subset \mathrm{Hom}(W,U)$) and let $v\in V$ be homogeneous. Then for $n\in \mathbb{N},\ w\in W$, by changing variable and using Lemma 3.5 we get

$$\begin{aligned}
&\mathrm{Res}_x x^{\mathrm{wt}v+n}(x-z)^{\mathrm{wt}v}fY^o(v,x)w\\
=\;&\mathrm{Res}_x x^{\mathrm{wt}v+n}(x-z)^{\mathrm{wt}v}fY(e^{xL(1)}(-x^{-2})^{L(0)}v,x^{-1})w\\
=\;&\mathrm{Res}_x x^{-\mathrm{wt}v-n-2}(x^{-1}-z)^{\mathrm{wt}v}fY(e^{x^{-1}L(1)}(-x^{2})^{L(0)}v,x)w\\
=\;&(-1)^{\mathrm{wt}v}f\left(\mathrm{Res}_x x^{-n-2}(1-zx)^{\mathrm{wt}v}Y(e^{x^{-1}L(1)}v,x)w\right)\\
=\;&0.
\end{aligned} \tag{3.25}$$

This shows

$$x^{\mathrm{wt}v}(x-z)^{\mathrm{wt}v}fY^o(v,x)\in (\mathrm{Hom}(W,U))[[x]]. \tag{3.26}$$

That is, $f\in T$. Thus

$$\mathrm{Hom}(A(W,z),U)\subset T.$$

From the definition of $\mathcal{D}_{P(z)}(W,U)$, we immediately have

$$T\subset \mathcal{D}_{P(z)}(W,U).$$

Let $f\in T$ and let $v\in V$ be homogeneous. By Lemma 2.13 we have

$$(-z+x)^{\mathrm{wt}v}Y^R(v,x)f=(x-z)^{\mathrm{wt}v}fY^o(v,x), \tag{3.27}$$
$$(z+x)^{\mathrm{wt}v}Y^L(v,x)f=(x+z)^{\mathrm{wt}v}fY^o(v,x+z). \tag{3.28}$$

Then

$$x^{\mathrm{wt}v}(-z+x)^{\mathrm{wt}v}Y^R(v,x)f=x^{\mathrm{wt}v}(x-z)^{\mathrm{wt}v}fY^o(v,x)\in (\mathrm{Hom}(W,U))[[x]], \tag{3.29}$$
$$x^{\mathrm{wt}v}(z+x)^{\mathrm{wt}v}Y^L(v,x)f=x^{\mathrm{wt}v}(x+z)^{\mathrm{wt}v}fY^o(v,x+z)\in (\mathrm{Hom}(W,U))[[x]]. \tag{3.30}$$

We are also using (2.33). Hence

$$x^{\mathrm{wt}v}Y^R(v,x)f=(-z+x)^{-\mathrm{wt}v}\left[x^{\mathrm{wt}v}(-z+x)^{\mathrm{wt}v}Y^R(v,x)f\right]\in (\mathrm{Hom}(W,U))[[x]], \tag{3.31}$$
$$x^{\mathrm{wt}v}Y^L(v,x)\alpha=(z+x)^{-\mathrm{wt}v}\left[x^{\mathrm{wt}v}(z+x)^{\mathrm{wt}v}Y^L(v,x)f\right]\in (\mathrm{Hom}(W,U))[[x]]. \tag{3.32}$$

It follows from Lemma 3.7 that $f \in \Omega(\mathcal{D}_{P(z)}(W,U))$. This proves

$$T \subset \Omega(\mathcal{D}_{P(z)}(W,U)).$$

Let $f \in \Omega(\mathcal{D}_{P(z)}(W,U))$ and let $v \in V$ be homogeneous. Then

$$x^{\mathrm{wt}\, v} Y^L(v,x)f, \quad x^{\mathrm{wt}\, v} Y^R(v,x)f \in (\mathrm{Hom}(W,U))[[x]]. \tag{3.33}$$

Multiplying (2.59) by $x^{\mathrm{wt}\, v} x_0^{\mathrm{wt}\, v}$, then taking Res_{x_0} (and using the fundamental properties of delta functions) we get

$$\begin{aligned} & x^{\mathrm{wt}\, v}(x-z)^{\mathrm{wt}\, v} f Y^o(v,x) - x^{\mathrm{wt}\, v}(-z+x)^{\mathrm{wt}\, v} Y^R(v,x)f \\ = \ & \mathrm{Res}_{x_0} z^{-1}\delta\left(\frac{x-x_0}{z}\right)(z+x_0)^{\mathrm{wt}\, v} x_0^{\mathrm{wt}\, v} Y^L(v,x_0)f. \end{aligned} \tag{3.34}$$

Then it follows from (3.33) that

$$x^{\mathrm{wt}\, v}(x-z)^{\mathrm{wt}\, v} f Y^o(v,x) = x^{\mathrm{wt}\, v}(-z+x)^{\mathrm{wt}\, v} Y^R(v,x)f \in (\mathrm{Hom}(W,U))[[x]]. \tag{3.35}$$

That is, $f \in T$. Furthermore, for homogeneous $v \in V$ and for $w \in W$, since $(Y^o)^o = Y$ [FHL], we have

$$\begin{aligned} & \mathrm{Res}_x x^{-2}(1-zx)^{\mathrm{wt}\, v} f Y(v,x)w \\ = \ & \mathrm{Res}_x x^{-2}(1-zx)^{\mathrm{wt}\, v} f Y^o(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w \\ = \ & \mathrm{Res}_x (1-zx^{-1})^{\mathrm{wt}\, v} f Y^o(e^{x^{-1}L(1)}(-x^{2})^{L(0)}v, x)w \\ = \ & \mathrm{Res}_x (-1)^{\mathrm{wt}\, v} x^{\mathrm{wt}\, v}(x-z)^{\mathrm{wt}\, v} f Y^o(e^{x^{-1}L(1)}v, x)w \\ = \ & \sum_{i\geq 0} (-1)^{\mathrm{wt}\, v}\frac{1}{i!}\mathrm{Res}_x x^{\mathrm{wt}\, v - i}(x-z)^{\mathrm{wt}\, v} f Y^o(L(1)^i v, x)w \\ = \ & \sum_{i\geq 0} (-1)^{\mathrm{wt}\, v}\frac{1}{i!}\mathrm{Res}_x x^{\mathrm{wt}(L(1)^i v)}(x-z)^{\mathrm{wt}(L(1)^i v)+i} f Y^o(L(1)^i v, x)w \\ = \ & 0 \end{aligned} \tag{3.36}$$

because

$$\begin{aligned} & \mathrm{Res}_x x^{\mathrm{wt}(L(1)^i v)}(x-z)^{\mathrm{wt}(L(1)^i v)+i} f Y^o(L(1)^i v, x)w \\ = \ & \sum_{j=0}^{i}\binom{i}{j}\mathrm{Res}_x x^{\mathrm{wt}(L(1)^i v)+j}(x-z)^{\mathrm{wt}(L(1)^i v)} f Y^o(L(1)^i v, x)w \\ = \ & 0. \end{aligned} \tag{3.37}$$

This proves $f(O(W,z)) = 0$, hence $f \in \mathrm{Hom}(A(W,z),U)$. Thus $\Omega(\mathcal{D}_{P(z)}(W,U)) \subset \mathrm{Hom}(A(W,z),U)$. This completes the proof. $\square$

It follows from Theorem 2.17, Lemma 3.7, and Proposition 3.2 that $\Omega(\mathcal{D}_{P(z)}(W,U))$ is an $A(V)\otimes A(V)$-module. On the other hand, because $A(W,z)$ is an $A(V)$-bimodule

and θ is an involution of $A(V)$, from the classical fact $\mathrm{Hom}(A(W,z),U)$ becomes an $A(V)\otimes A(V)$-module with

$$((a_1,a_2)f)(w)=f(\theta(a_2)wa_1) \tag{3.38}$$

for $a_1,a_2\in A(V),\ f\in \mathrm{Hom}(A(W,z),U),\ w\in A(W,z)$.

Strengthening Proposition 3.8 we have:

Theorem 3.9 *Let W be a weak V-module, U a vector space and z a nonzero complex number. With the above defined $A(V)\otimes A(V)$-module structures,* $\mathrm{Hom}(A(W,z),U)$ *and* $\Omega(\mathcal{D}_{P(z)}(W,U))$ *coincide.*

Proof. Let $f\in \mathrm{Hom}(A(W,z),U)$ and let $v\in V$ be homogeneous. From Proposition 3.8, we have

$$x^{\mathrm{wt}v}Y^L(v,x)f,\ \ x^{\mathrm{wt}v}Y^R(v,x)f\in \mathcal{D}_{P(z)}(W,U)[[x]],$$

Then by expanding $(-z+x)^{\mathrm{wt}v}$ and $(z+x)^{\mathrm{wt}v}$ we get

$$\mathrm{Res}_x x^{\mathrm{wt}v-1}Y^R(v,x)f=\mathrm{Res}_x(-z)^{-\mathrm{wt}v}x^{\mathrm{wt}v-1}(-z+x)^{\mathrm{wt}v}Y^R(v,x)f, \tag{3.39}$$
$$\mathrm{Res}_x x^{\mathrm{wt}v-1}Y^L(v,x)f=\mathrm{Res}_x z^{-\mathrm{wt}v}x^{\mathrm{wt}v-1}(z+x)^{\mathrm{wt}v}Y^L(v,x)f. \tag{3.40}$$

Then for $w\in W$, using (3.23) we have

$$\begin{aligned}
&\mathrm{Res}_x x^{\mathrm{wt}v-1}(Y^R(v,x)f)(w)\\
=\ &\mathrm{Res}_x(-z)^{-\mathrm{wt}v}x^{\mathrm{wt}v-1}(-z+x)^{\mathrm{wt}v}(Y^R(v,x)f)(w)\\
=\ &\mathrm{Res}_x(-z)^{-\mathrm{wt}v}x^{\mathrm{wt}v-1}(x-z)^{\mathrm{wt}v}fY^o(v,x)w\\
=\ &\mathrm{Res}_x(-z)^{-\mathrm{wt}v}x^{\mathrm{wt}v-1}(x-z)^{\mathrm{wt}v}fY(e^{xL(1)}(-x^{-2})^{L(0)}v,x^{-1})w\\
=\ &\mathrm{Res}_x(-1)^{\mathrm{wt}v}(-z)^{-\mathrm{wt}v}x^{-\mathrm{wt}v-1}(x-z)^{\mathrm{wt}v}fY(e^{xL(1)}v,x^{-1})w\\
=\ &\mathrm{Res}_x(-1)^{\mathrm{wt}v}(-z)^{-\mathrm{wt}v}x^{-1}(1-zx)^{\mathrm{wt}v}fY(e^{x^{-1}L(1)}v,x)w\\
=\ &\sum_{i\geq 0}\frac{1}{i!}\mathrm{Res}_x(-1)^{\mathrm{wt}v}(-z)^{-\mathrm{wt}v}x^{-1-i}(1-zx)^{\mathrm{wt}(L(1)^iv)+i}fY(L(1)^iv,x)w\\
=\ &\sum_{i,j\geq 0}\frac{1}{i!}\binom{i}{j}\mathrm{Res}_x(-1)^{\mathrm{wt}v}(-z)^{-\mathrm{wt}v+j}x^{-1-i+j}(1-zx)^{\mathrm{wt}(L(1)^iv)}fY(L(1)^iv,x)w\\
=\ &\sum_{i\geq 0}\frac{1}{i!}\mathrm{Res}_x(-z)^{-\mathrm{wt}v+i}x^{-1}(1-zx)^{\mathrm{wt}(L(1)^iv)}fY(L(1)^i(-1)^{L(0)}v,x)w\\
=\ &f(\theta(v)*_{P(z)}w).
\end{aligned} \tag{3.41}$$

Here we are using the fact:

$$\mathrm{Res}_x x^{-1-r}(1-zx)^{\mathrm{wt}(L(1)^iv)}Y(L(1)^iv,x)w\in O(W,z)$$

for $r\geq 1$ (Lemma 3.5).

Similarly, using (3.24) we get

$$
\begin{aligned}
& \mathrm{Res}_x x^{\mathrm{wt} v-1}(Y^L(v,x)f)(w) \\
= \; & \mathrm{Res}_x z^{-\mathrm{wt} v} x^{\mathrm{wt} v-1}(z+x)^{\mathrm{wt} v}(Y^L(v,x)f)(w) \\
= \; & \mathrm{Res}_x z^{-\mathrm{wt} v} x^{\mathrm{wt} v-1}(x+z)^{\mathrm{wt} v} f Y^o(v,x+z)w \\
= \; & \mathrm{Res}_x z^{-\mathrm{wt} v}(x-z)^{\mathrm{wt} v-1} x^{\mathrm{wt} v} f Y^o(v,x)w \\
= \; & \mathrm{Res}_x z^{-\mathrm{wt} v}(x-z)^{\mathrm{wt} v-1} x^{\mathrm{wt} v} f Y(e^{xL(1)}(-x^{-2})^{L(0)}v,x^{-1})w \\
= \; & \mathrm{Res}_x (-z)^{-\mathrm{wt} v}(1-zx)^{\mathrm{wt} v-1} x^{-1} f Y(e^{x^{-1}L(1)}v,x)w \\
= \; & \sum_{i\geq 0}\frac{1}{i!}\mathrm{Res}_x(-z)^{-\mathrm{wt} v} x^{-1-i}(1-zx)^{\mathrm{wt}(L(1)^i v)+i-1} f Y(L(1)^i v,x)w \\
= \; & \mathrm{Res}_x(-z)^{-\mathrm{wt} v} x^{-1}(1-zx)^{\mathrm{wt} v-1} f Y(v,x)w \\
& +\sum_{i\geq 1}\frac{1}{i!}\mathrm{Res}_x(-z)^{-\mathrm{wt} v} x^{-2-(i-1)}(1-zx)^{\mathrm{wt}(L(1)^i v)+i-1} f Y(L(1)^i v,x)w \\
= \; & \mathrm{Res}_x(-z)^{-\mathrm{wt} v} x^{-1}(1-zx)^{\mathrm{wt} v-1} f Y(v,x)w \\
= \; & f(w *_{P(z)} v)
\end{aligned}
\tag{3.42}
$$

because for $i\geq 1$,

$$\mathrm{Res}_x(-1)^{\mathrm{wt} v} x^{-2-(i-1)}(1-zx)^{\mathrm{wt}(L(1)^i v)+i-1} Y(L(1)^i v,x)w \in O(W,z).$$

Then it follows immediately from the definitions of the module structures. $\square$

Recall that $\mathrm{Hom}(A(W,z),U)$ is an $A(V)\otimes A(V)$-module with the action defined in (3.38). Now, *let U be a (left) $A(V)$-module instead of just a vector space and let $z=-1$.* Then $\mathrm{Hom}_{A(V)}(A(V),U)$ is a (left) $A(V)$-submodule of $\mathrm{Hom}(A(V),U)$ equipped with the first action of $A(V)$ (recall (3.38)), i.e.,

$$(af)(b) = f(ba) \tag{3.43}$$

for $a,b\in A(V),\ f\in \mathrm{Hom}(A(V),U)$. Furthermore, as an $A(V)$-module,

$$U = \mathrm{Hom}_{A(V)}(A(V),U). \tag{3.44}$$

Definition 3.10 Let U be an $A(V)$-module. We define $\mathrm{Ind}_{A(V)}^V U$ to be the V-submodule of $(\mathcal{D}_{P(-1)}(W,U), Y^L_{P(-1)})$, generated by U $(=\mathrm{Hom}_{A(V)}(A(V),U))$.

We shall briefly use $\mathrm{Ind}\, U$ for $\mathrm{Ind}_{A(V)}^V U$ whenever it is clear from the context.

Lemma 3.11 *Let U be a (left) $A(V)$-module. Then*

$$U = \mathrm{Hom}_{A(V)}(A(V),U) \subset \Omega(\mathrm{Ind}\, U) \subset \mathrm{Hom}(A(V),U). \tag{3.45}$$

Proof. Because

$$U = \mathrm{Hom}_{A(V)}(A(V), U) \subset \mathrm{Hom}(A(V), U) = \Omega_{V\otimes V}(\mathcal{D}_{P(-1)}(V, U)), \tag{3.46}$$

and the actions Y^L and Y^R commute, we have

$$Y^L(v, x)U \subset (\Omega_V(\mathcal{D}_{P(-1)}(V, U), Y^R))[[x]] \tag{3.47}$$

for $v \in V$. Furthermore, because U generates Ind U under the action Y^L, we have

$$\mathrm{Ind}\ U \subset \Omega_V(\mathcal{D}_{P(-1)}(V, U), Y^R). \tag{3.48}$$

Then using Theorem 3.9, we get

$$\Omega(\mathrm{Ind}\ U) \subset \Omega_V\left(\Omega_V(\mathcal{D}_{P(-1)}(V, U), Y^R), Y^L\right) = \Omega_{V\otimes V}(\mathcal{D}_{P(-1)}(V, U)) = \mathrm{Hom}(A(V), U) \tag{3.49}$$

This completes the proof. $\square$

Let U_1 and U_2 be $A(V)$-modules and left ψ be an $A(V)$-homomorphism from U_1 to U_2. Then f gives rise to a homomorphism f^o from $\mathrm{Hom}(V, U_1)$ to $\mathrm{Hom}(V, U_2)$ in the obvious way. Furthermore, it is easy to see that the restriction of f^o is a $V \otimes V$-homomorphism from $\mathcal{D}_{P(-1)}(V, U_1)$ to $\mathcal{D}_{P(-1)}(V, U_2)$, which maps $\mathrm{Hom}(A(V), U_1)$ to $\mathrm{Hom}(A(V), U_2)$. The restriction of f^o to Ind U_1 is a V-homomorphism from Ind U_1 to Ind U_2. It is routine to check that the map Ind : $U \mapsto$ Ind U gives rise to a functor from the category of $A(V)$-modules to the category of weak V-modules. It is also clear that

$$\mathrm{Ind}(U_1 \oplus U_2) = \mathrm{Ind}\ U_1 \oplus \mathrm{Ind}\ U_2. \tag{3.50}$$

Next we study the structure of the induced module Ind U. First, we prove the following result (cf. Lemma 2.1), which is a reformulation of a result of [DLM3]:

Lemma 3.12 *Let W be a weak V-module, $w \in W$. Let $u, v \in V$ and let $k \in \mathbb{Z}$ be such that*

$$x^k Y(u, x)w \in W[[x]], \tag{3.51}$$

or equivalently,

$$u_{k+m}w = 0 \quad \text{for } m \geq 0. \tag{3.52}$$

Then for $p, q \in \mathbb{Z}$,

$$u_p v_q w = \sum_{i=0}^{n} \sum_{j\geq 0} \binom{p-k}{i} \binom{k}{j} (u_{p-k-i+j}v)_{q+k+i-j}w. \tag{3.53}$$

where n is any nonnegative integer such that $x^{n+1+q}Y(v, x)w \in W[[x]]$.

Proof. Since $x^k Y(u,x)w \in W[[x]]$, by applying $\mathrm{Res}_{x_1} x_1^k$ to the Jacobi identity for the triple (u,v,w) we get [DL]

$$(x_0+x_2)^k Y(u,x_0+x_2)Y(v,x_2)w = (x_2+x_0)^k Y(Y(u,x_0)v,x_2)w. \tag{3.54}$$

Notice that

$$u_p v_q w = \mathrm{Res}_{x_0}\mathrm{Res}_{x_2} x_2^q (x_0+x_2)^p Y(u,x_0+x_2)Y(v,x_2)w. \tag{3.55}$$

We can multiply the left hand side of (3.54) by $x_2^q(x_0+x_2)^{p-k}$, but we are not allowed to multiply the right hand side of (3.54) by $x_2^q(x_0+x_2)^{p-k}$. Notice that

$$(x_0+x_2)^{p-k} = \sum_{i=0}^{n} \binom{p-k}{i} x_0^{p-k-i} x_2^i + \sum_{i\geq n+1} \binom{p-k}{i} x_0^{p-k-i} x_2^i \tag{3.56}$$

and

$$\mathrm{Res}_{x_2} \sum_{i\geq n+1} \binom{p-k}{i} x_0^{p-k-i} x_2^{i+q} (x_0+x_2)^k Y(u,x_0+x_2)Y(v,x_2)w = 0 \tag{3.57}$$

because $x^{n+1+q} Y(v,x)w \in W[[x]]$. Using (3.54)-(3.57) we get

$$\begin{aligned}
& u_p v_q w \\
= \; & \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}(x_0+x_2)^p x_2^q Y(u,x_0+x_2)Y(v,x_2)w \\
= \; & \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}(x_0+x_2)^{p-k} x_2^q \left((x_0+x_2)^k Y(u,x_0+x_2)Y(v,x_2)w\right) \\
= \; & \mathrm{Res}_{x_0}\mathrm{Res}_{x_2} \sum_{i=0}^{n} \binom{p-k}{i} x_0^{p-k-i} x_2^{i+q} \left((x_0+x_2)^k Y(u,x_0+x_2)Y(v,x_2)w\right) \\
= \; & \mathrm{Res}_{x_0}\mathrm{Res}_{x_2} \sum_{i=0}^{n} \binom{p-k}{i} x_0^{p-k-i} x_2^{i+q} \left((x_2+x_0)^k Y(Y(u,x_0)v,x_2)w\right) \\
= \; & \sum_{i=0}^{n}\sum_{j\geq 0} \binom{p-k}{i}\binom{k}{j} (u_{p-k-i+j}v)_{q+k+i-j}w.
\end{aligned} \tag{3.58}$$

This concludes the proof. $\square$

As an immediate consequence of Lemma 3.12, we have the following result, which was proved in [DM] and [Li1].

Corollary 3.13 *Let W be a weak V-module and let $w \in W$. Set*

$$\langle w \rangle = \text{linear span } \{v_m w \mid v \in V,\ m \in \mathbb{Z}\}. \tag{3.59}$$

Then $\langle w \rangle$ is the sub-weak-module of W, generated by w.

Furthermore, we have:

Lemma 3.14 *Let W be a weak V-module and let U be an irreducible $A(V)$-submodule of $\Omega(W)$. Then the weak submodule $\langle U\rangle$ of W, generated by U, is a lowest weight generalized V-module with U being the lowest weight subspace.*

Proof. From Corollary 3.13 we have

$$\langle U\rangle = \text{linear span}\{v_m U \mid v \in V,\ m \in \mathbb{Z}\}. \tag{3.60}$$

Since V has a countable basis, $A(V)$, being a quotient of V, also has a countable basis. Then the central element $\omega + O(V)$ of $A(V)$ acts on any irreducible $A(V)$-module as a scalar. (See the proof of Lemma 1.2.1, [Z].) That is, $L(0)$ acts as a scalar h on U (being a subspace of W). Then it immediately follows from (3.60) and the following facts:

$$\mathrm{wt} v_m = \mathrm{wt} v - m - 1, \tag{3.61}$$

$$v_n U = 0 \tag{3.62}$$

for homogeneous $v \in V$ and for $m \in \mathbb{Z},\ n \geq \mathrm{wt} v$. $\square$

We immediately have:

Corollary 3.15 *Let U be an irreducible (left) $A(V)$-module. Then* Ind U *is a lowest weight generalized V-module with U as the lowest weight subspace.* $\square$

Remark 3.16 There are two questions regarding Ind U: (1) Is $\mathrm{Ind}_{A(V)}^{V} U$ already an irreducible generalized V-module? (2) Is there a canonical characterization of Ind U?

4 Functor F and Frenkel-Zhu's fusion rule theorem

The main goal of this section is to give an alternate proof of Frenkel and Zhu's fusion rule theorem.

Recall from [B] (cf. [FFR], [Li1]) the Lie algebra $g(V)$ associated to the vertex operator algebra V. As a vector space,

$$g(V) = \hat{V}/D\hat{V},$$

where

$$\hat{V} = V \otimes \mathbb{C}[t, t^{-1}], \quad D = L(-1) \otimes 1 + 1 \otimes \frac{d}{dt}.$$

The Lie bracket is given by

$$[u(m), v(n)] = \sum_{i \geq 0} \binom{m}{i} (u_i v)(m + n - i)$$

for $u, v \in V,\ m, n \in \mathbb{Z}$, where $u(m) = u \otimes t^m$. Furthermore, $g(V)$ is naturally a $\mathbb{Z}$-graded Lie algebra with

$$\deg v(m) = \mathrm{wt} v - m - 1 \tag{4.1}$$

for homogeneous $v \in V$ and for $m \in \mathbb{Z}$. It is clear that any weak V-module is a natural $g(V)$-module and that any generalized V-module is a $\mathbb{C}$-graded $g(V)$-module. It was known (cf. [Li1]) that $A(V)_{Lie}$ is a natural quotient Lie algebra of $g(V)_0$, where $g(V)_0$ is the degree-zero Lie subalgebra of $g(V)$. Then any $A(V)$-module is a natural $g(V)_0$-module.

Recall a notion from [DLM2]. (Here we use a different symbol for the universal object.)

Definition 4.1 Let U be an $A(V)$-module. Then U is a $g(V)_0$-module. View U as a $(g(V)_0 + g(V)_-)$-module with $g(V)_- U = 0$, where $g(V)_- = \oplus_{n>0} g(V)_{-n}$. Define the standard induced $g(V)$-module

$$\tilde{F}(U) = U(g(V)) \otimes_{U(g(V)_- + g(V)_0)} U, \tag{4.2}$$

which is an $\mathbb{N}$-graded $g(V)$-module with

$$\deg U = 0. \tag{4.3}$$

Then we define $F(U)$ to be the quotient $g(V)$-module of $\tilde{F}(U)$ modulo the following Jacobi identity relation:

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)Y(v, x_2)w - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2)Y(u, x_1)w \\ = \; & x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2)w \end{aligned} \tag{4.4}$$

for $u, v \in V,\ w \in \tilde{F}(U)$.

From definition, $F(U)$ is an $\mathbb{N}$-graded $g(V)$-module. Because of (4.4), $F(U)$ clearly is an $\mathbb{N}$-graded weak V-module. Let e_U be the natural map from U to $F(U)$. Then we have the following obvious universal property:

Proposition 4.2 *Let W be any weak V-module and let ψ be any $A(V)$-homomorphism from U to $\Omega(W)$. Then there exists a unique V-homomorphism $\tilde{\psi}$ from $F(U)$ to W such that $\tilde{\psi} e_U = \psi$.* $\square$

Note that we have not excluded the possibility that $F(U) = 0$ even if $U \neq 0$. With the weak V-module Ind U we have the following result:

Lemma 4.3 *Let U be an $A(V)$-module. Then the natural linear map e_U from U to $F(U)$ is injective and $e_U(U) = F(U)(0)$.*

Proof. It is clear that $e_U(U) = F(U)(0)$. Since U is an $A(V)$-submodule of $\Omega(\text{Ind } U)$, using the universal property of $F(U)$ (Proposition 4.2), we obtain a V-homomorphism ϕ from $F(U)$ to Ind U such that ϕe_U is the embedding of U into $\Omega(\text{Ind } U)$. In particular, ϕe_U is injective. Consequently, e_U is injective. $\square$

In view of Lemma 4.3, we consider U as a canonical subspace of $F(U)$. Combining Lemma 4.3 with Lemma 3.14 we immediately have:

Lemma 4.4 *Let U be an irreducible $A(V)$-module. Then $F(U)$ is a lowest weight generalized V-module with U as the lowest weight subspace.* $\square$

Consider all graded submodules W of $F(U)$ such that $W \cap U = 0$. Then the sum of all such graded submodules is still a such graded submodule, so that it is the unique maximal graded submodule with this property. Define $L(U)$ to be the quotient module of $F(U)$ modulo the maximal submodule. We have ([Z], Theorem 2.2.1):

Lemma 4.5 *Let U be an $A(V)$-module. Then $L(U)$ is an $\mathbb{N}$-graded weak V-module such that for any nonzero graded submodule W of $L(U)$, $U \cap W \neq 0$. Furthermore, if U is irreducible, $L(U)$ is an irreducible generalized V-module.*

Proof. The first assertion directly follows from the definition of $L(U)$. Since U is irreducible, by Lemma 3.14, $L(U)$ is a lowest weight generalized V-module with U as the lowest weight subspace. Then the $\mathbb{N}$-grading on $L(U)$ is a shift of the $L(0)$-grading on W. Consequently, any submodule of $L(U)$ is automatically graded. It follows immediately that $L(U)$ is irreducible. $\square$

It is routine to check that the map $F : U \mapsto F(U)$ gives rise to a functor F from the category of $A(V)$-modules to the category of $\mathbb{N}$-graded weak V-modules. Furthermore, given a family of $A(V)$-modules U_i for $i \in S$, we have

$$F(\oplus_{i\in S} U_i) = \oplus_{i\in S} F(U_i), \tag{4.5}$$

or equivalently, if U is an $A(V)$-module such that $U = E \otimes U_1$ where E is a vector space and U_1 is an $A(V)$-module, then $F(U) = E \otimes F(U_1)$. We also have the following analogue of the Frobenius reciprocity theorem (cf. [Ki]):

Lemma 4.6 *Let W be a weak V-module and let U be an $A(V)$-module. Then the map*

$$\begin{aligned} \Omega' : \quad & \mathrm{Hom}_V(F(U), W) \to \mathrm{Hom}_{A(V)}(U, \Omega(W)) \\ & \psi \mapsto \Omega(\psi) \end{aligned} \tag{4.6}$$

is a linear isomorphism.

Proof. Because U generates $F(U)$ as a weak V-module, it is clear that Ω' is injective. It follows from the universal property of $F(U)$ (Proposition 4.2) that Ω' is also surjective. $\square$

Remark 4.7 Let W and U be given as in Lemma 4.6. Similarly, we define a linear map Ω'' from $\mathrm{Hom}_V(\mathrm{Ind}\, U, W)$ to $\mathrm{Hom}_{A(V)}(U, \Omega(W))$. Then Ω'' is injective. It is easy to see that Ω'' is surjective if and only if the V-homomorphism from $F(U)$ to $\mathrm{Ind}\, U$, extending the identity map of U, is an isomorphism.

We shall need the following fact:

Lemma 4.8 *Let V_1 and V_2 be vertex operator algebras and let U_1 and U_2 be $A(V_1)$ and $A(V_2)$-modules, respectively. Let W be a weak $V_1 \otimes V_2$-module and let ψ be an $A(V_1) \otimes A(V_2)$-homomorphism from $U_1 \otimes U_2$ to $\Omega(W)$. Then there exists a unique $V_1 \otimes V_2$-homomorphism $\bar{\psi}$ from $F(U_1) \otimes F(U_2)$ to W, extending ψ.*

Proof. The uniqueness is clear because $U_1 \otimes U_2$ generates $F(U_1) \otimes F(U_2)$ as a weak $V_1 \otimes V_2$-module. By Proposition 4.2, there exists a V_1-homomorphism ψ_1 from $F_{V_1}(U_1 \otimes U_2)$ to W, extending ψ. Note that

$$F_{V_1}(U_1) \otimes U_2 = F_{V_1}(U_1 \otimes U_2).$$

It is clear that ψ_1 is an $A(V_2)$-homomorphism. Then by Proposition 4.2 again, there exists a V_2-homomorphism ψ_2 from

$$F_{V_2}(F(U_1) \otimes U_2) \ \ (= F_{V_1}(U_1) \otimes F_{V_2}(U_2))$$

to W, extending ψ_1. Consequently, ψ_2 is a $V_1 \otimes V_2$-homomorphism $\bar{\psi}$ from $F(U_1) \otimes F(U_2)$ to W, extending ψ. $\square$

Remark 4.9 Let V_1, V_2, U_1 and U_2 be given as in Lemma 4.8. It was proved in [DMZ] that $A(V_1 \otimes V_2)$ is naturally isomorphic to $A(V_1) \otimes A(V_2)$. Then $U_1 \otimes U_2$ is a natural $A(V_1 \otimes V_2)$-module. By Lemma 4.8, there exists a $V_1 \otimes V_2$-homomorphism ψ from $F(U_1) \otimes F(U_2)$ to $F_{V_1 \otimes V_2}(U_1 \otimes U_2)$, extending the identity map of $U_1 \otimes U_2$. It follows from the universal property of $F_{V_1 \otimes V_2}(U_1 \otimes U_2)$ that ψ is an isomorphism.

Recall the involution (anti-automorphism) θ of $A(V)$. Let U be a (left) $A(V)$-module. Then from the classical fact U^* is a left $A(V)$-module with the action defined by

$$(af)(u) = f(\theta(a)u) \quad \text{for } a \in A(V),\ u \in U. \tag{4.7}$$

The following result is classical in nature.

Lemma 4.10 *Let U_1, U_2 be (left) $A(V)$-modules and let B be an $A(V)$-bimodule. Define*

$$\begin{aligned} d: \quad & \mathrm{Hom}(B \otimes_{A(V)} U_1, U_2) \to \mathrm{Hom}(U_1 \otimes U_2^*, B^*) \\ & \psi \mapsto d_\psi, \end{aligned} \tag{4.8}$$

where for $u_1 \in U_1,\ u_2^* \in U_2^*$,

$$\langle d_\psi(u_1 \otimes u_2^*), b\rangle = \langle u_2^*, \psi(b \otimes u_1)\rangle. \tag{4.9}$$

Then

$$d\left(\mathrm{Hom}_{A(V)}(B \otimes_{A(V)} U_1, U_2)\right) \subset \mathrm{Hom}_{A(V)\otimes A(V)}(U_1 \otimes U_2^*, B^*), \tag{4.10}$$

where B^* is considered as an $A(V) \otimes A(V)$-module with the action defined by (3.38). If we in addition assume that U_2 is finite-dimensional, then the restriction of d gives rise to a linear isomorphism from $\mathrm{Hom}_{A(V)}(B \otimes_{A(V)} U_1, U_2)$ onto $\mathrm{Hom}_{A(V)\otimes A(V)}(U_1 \otimes U_2^*, B^*)$.

In particular, let U be a (left) $A(V)$-module. Then the map

$$\begin{aligned} d_U: \quad & U \otimes U^* \to A(V)^* \\ & u \otimes u^* \mapsto d_U(u \otimes u^*), \end{aligned} \tag{4.11}$$

where for $a \in A(V)$,

$$\langle d_U(u \otimes u^*), a\rangle = \langle u^*, au\rangle, \tag{4.12}$$

is an $A(V) \otimes A(V)$-homomorphism.

Proof. Let η' be the natural embedding of $\mathrm{Hom}(B \otimes U_1, U_2)$ into $\mathrm{Hom}(U_1 \otimes U_2^*, B^*)$. It is a classical fact that η' is a linear isomorphism if U_2 is finite-dimensional. With $B \otimes_{A(V)} U_1$ being a quotient space of $B \otimes U_1$, we naturally consider $\mathrm{Hom}(B \otimes_{A(V)} U_1, U_2)$ as a subspace of $\mathrm{Hom}(B \otimes U_1, U_2)$. Then d is the restriction of η'. Thus d is injective. Let $\psi \in \mathrm{Hom}(B \otimes U_1, U_2),\ a_1, a_2 \in A(V),\ u_1 \in U_1,\ u_2^* \in U_2^*$ and $b \in B$. Then

$$\begin{aligned} \langle d_\psi((a_1, a_2)(u_1 \otimes u_2^*)), b\rangle &= \langle d_\psi(a_1u_1 \otimes a_2u_2^*), b\rangle \\ &= \langle a_2u_2^*, \psi(b \otimes a_1u_1)\rangle \\ &= \langle u_2^*, \theta(a_2)\psi(b \otimes a_1u_1)\rangle \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} & \langle (a_1, a_2)d_\psi(u_1 \otimes u_2^*), b\rangle\rangle \\ = \ & \langle d_\psi(u_1 \otimes u_2^*), \theta(a_2)ba_1\rangle \\ = \ & \langle u_2^*, \psi(\theta(a_2)ba_1 \otimes u_1)\rangle. \end{aligned} \tag{4.14}$$

It follows immediately that $\psi \in \mathrm{Hom}_{A(V)}(B \otimes_{A(V)} U_1, U_2)$ if and only if

$$d_\psi \in \mathrm{Hom}_{A(V)\otimes A(V)}(U_1 \otimes U_2^*, B^*).$$

This completes the proof. $\square$

Remark 4.11 Let U be an irreducible (left) $A(V)$-module. Let u^* be a nonzero element of U^*. By Lemma 4.10, $d_U(\cdot \otimes u^*)$ gives an $A(V)$-homomorphism from U to $A(V)^*$ equipped with the first action. It follows from the irreducibility of U that $d_U(\cdot \otimes u^*)$ is injective. The it follows from Lemma 3.14 that there is a canonical lowest weight generalized V-module inside the regular representation on $\mathcal{D}_{P(-1)}(V)$ with U as the lowest weight subspace.

Lemma 4.12 *Let W be a weak V-module and let W_1 and W_2 be lowest weight generalized V-modules with lowest weight subspaces $W_1(0)$ and $W_2(0)$, respectively. Define the restriction map*

$$\begin{aligned} \Omega_T: \quad & \mathrm{Hom}_{V\otimes V}(W_1 \otimes W_2, \mathcal{D}_{P(-1)}(W)) \to \mathrm{Hom}_{A(V)\otimes A(V)}(W_1(0) \otimes W_2(0), \Omega(\mathcal{D}_{P(-1)}(W))) \\ & \psi \mapsto \Omega(\psi)|_{W_1(0)\otimes W_2(0)}. \end{aligned} \tag{4.15}$$

Then Ω_T is injective.

Proof. It immediately follows from the fact that the $V \otimes V$-module $W_1 \otimes W_2$ is generated by $W_1(0) \otimes W_2(0)$. $\square$

Recall from Theorem 2.9 that for generalized V-modules W, W_1 and W_2, $F_p[P(z)]_{W_1W_2}^{W'}$ is a linear isomorphism from $\mathcal{V}_{W_1W_2}^{W'}$ onto $\mathrm{Hom}_{V\otimes V}(W_1 \otimes W_2, \mathcal{D}_{P(z)}(W))$. For the rest of this section, we use $F_{W_1W_2}^{W'}$ for $F_0[P(-1)]_{W_1W_2}^{W'}$.

Combining Theorem 2.9 and Lemma 4.12 with Theorem 3.9 we immediately have:

Proposition 4.13 *Let W, W_1 and W_2 be lowest weight generalized V-modules. Then $\Omega_T F_{W_1W_2}^{W'}$ is an injective linear map from $\mathcal{V}_{W_1W_2}^{W'}$ to $\mathrm{Hom}_{A(V)\otimes A(V)}(W_1(0)\otimes W_2(0), A(W)^*)$. In particular,*

$$\dim \mathcal{V}_{W_1W_2}^{W'} \leq \dim \mathrm{Hom}_{A(V)\otimes A(V)}(W_1(0) \otimes W_2(0), A(W)^*). \tag{4.16}$$

Next, we shall show that $\Omega_T F_{W_1W_2}^{W'}$ is a linear isomorphism in a certain situation.

Theorem 4.14 *Let W be a lowest weight generalized V-module and let U_1, U_2 be finite-dimensional irreducible $A(V)$-modules. Then the linear map*

$$\Omega_T F_{F(U_1)F(U_2)}^{W'}: \ \mathcal{V}_{F(U_1)F(U_2)}^{W'} \to \mathrm{Hom}_{A(V)\otimes A(V)}(U_1 \otimes U_2, A(W)^*)$$

is a linear isomorphism.

Proof. We only need to prove that $\Omega_T F_{F(U_1)F(U_2)}^{W'}$ is onto. For simplicity, in this proof we use $\mathcal{F}$ for $F_{F(U_1)F(U_2)}^{W'}$. Let

$$\psi \in \mathrm{Hom}_{A(V)\otimes A(V)}(U_1 \otimes U_2, A(W)^*).$$

Then $\psi(U_1 \otimes U_2)$ is an $A(V) \otimes A(V)$-submodule of $A(W)^*$, which is $\Omega(\mathcal{D}_{P(-1)}(W))$ by Proposition 3.8 with $U = \mathbb{C}$ and $z = -1$. By Lemma 4.4 we may assume that $\omega + O(V)$ acts as scalars h_1 and h_2 on U_1 and U_2, respectively. Then $L(0)$ acts as scalar $h_1 + h_2$ on $\psi(U_1 \otimes U_2)$. Let E be the $V \otimes V$-submodule of $\mathcal{D}_{P(-1)}(W)$, generated by $\psi(U_1 \otimes U_2)$. By Lemma 4.8, ψ extends to a $V \otimes V$-homomorphism $\bar{\psi}$ from $F(U_1) \otimes F(U_2)$ to E. By Theorem 2.9, we get an intertwining operator $\mathcal{F}^{-1}(\bar{\psi})$ of type $\binom{W'}{F(U_1)F(U_2)}$ such that

$$\Omega_T \mathcal{F}(\mathcal{F}^{-1}(\bar{\psi}) = \Omega(\bar{\psi}) = \psi.$$

Thus $\Omega_T \mathcal{F}$ is onto. This completes the proof. $\square$

Remark 4.15 In the proof, *if* E is an irreducible generalized $V \otimes V$-module, then from [FHL], $E = L(U_1) \otimes L(U_2)$, and then $\Omega\mathcal{F}$ will be a linear isomorphism from $\mathcal{V}_{L(U_1)L(U_2)}^{W'}$ to $\mathrm{Hom}_{A(V)\otimes A(V)}(U_1 \otimes U_2, A(W)^*)$. But, E in general is not irreducible.

Combining Theorem 4.14 with Lemma 4.10 we immediately have:

Corollary 4.16 *Let W, U_1 and U_2 be as in Theorem 4.14. Then $d^{-1}\Omega_T F_{F(U_1)F(U_2^*)}^{W'}$ is a linear isomorphism from $\mathcal{V}_{F(U_1)F(U_2^*)}^{W'}$ to $\mathrm{Hom}_{A(V)\otimes A(V)}(A(W) \otimes_{A(V)} U_1, U_2)$.* $\square$

Now we immediately have the following modified Frenkel-Zhu's fusion rule theorem (cf. [FZ], [Li1] and [Li2]):

Corollary 4.17 *Let W be a V-module and let W_1 and W_2 be irreducible V-modules such that $W_1 = F(W_1(0))$ and $W_2' = F((W_2(0))^*)$, or what is equivalent, $F(W_1(0))$ and $F((W_2(0))^*)$ are irreducible. Then*

$$\dim \mathrm{Hom}_{A(V)}(A(W) \otimes_{A(V)} W_1(0), W_2(0)) = \dim \mathcal{V}_{WW_1}^{W_2}. \tag{4.17}$$

In particular, this is true if V satisfies the condition that every lowest weight generalized V-module is completely reducible.

Proof. It was proved in [HL2] (cf. [FHL]) that

$$\dim \mathcal{V}_{WW_1}^{W_2} = \dim \mathcal{V}_{W_1W}^{W_2} = \dim \mathcal{V}_{W_1W_2'}^{W'}. \tag{4.18}$$

Then it follows immediately from Corollary 4.16. $\square$

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